

Hesitant Fuzzy Sets: State of the Art and Future Directions

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The necessity of dealing with uncertainty in real world problems has been a long-term research challenge that has originated different methodologies and theories. Fuzzy sets along with their extensions, such as type-2 fuzzy sets, interval-valued fuzzy sets, and Atanassov's intuitionistic fuzzy sets, have provided a wide range of tools that are able to deal with uncertainty in different types of problems. Recently, a new extension of fuzzy sets so-called hesitant fuzzy sets has been introduced to deal with hesitant situations, which were not well managed by the previous tools. Hesitant fuzzy sets have attracted very quickly the attention of many researchers that have proposed diverse extensions, several types of operators to compute with such types of information, and eventually some applications have been developed. Because of such a growth, this paper presents an overview on hesitant fuzzy sets with the aim of providing a clear perspective on the different concepts, tools and trends related to this extension of fuzzy sets. © 2014 Wiley Periodicals, Inc.

1. INTRODUCTION

A number of real-world problems are extremely complex, since they involve human beings and mechanical, technological, and other elements. Consequently, these problems and even simpler ones may have to consider diverse uncertainties whose handling is crucial for obtaining satisfactory solutions. Uncertainty can be considered of different types such as, randomness,¹ fuzziness,² indistinguishability,³ and incompleteness.⁴

Our interest is focused on fuzzy sets theory² and fuzzy logic that allow to manage imprecise and vague information. Such vagueness is reflected by the

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membership degree of the objects belonging to a concept.⁵ Fuzzy sets theory has been widely and successfully applied in many different areas to handle such a type of uncertainty. Nevertheless, it presents limitations to deal with imprecise and vague information when different sources of vagueness appear simultaneously. Owing to this fact and to overcome such limitations, different extensions of fuzzy sets have been introduced in the literature such as (i) *Atanassov's intuitionistic fuzzy sets* (IFS)⁶ that allow to simultaneously consider the membership degree and the nonmembership degree of each element, (ii) *type-2 fuzzy sets* (T2FS)⁷ that incorporate uncertainty in the definition of their membership function through the use of a fuzzy set over the unit interval to model it, (iii) *interval-valued fuzzy sets* (IVFS),^{8,9} in which the membership degree of an element is given by a closed subinterval of the unit interval in such a way that the length of that interval may be understood as a measure of the lack of certainty for building the precise membership degree of the element, (iv) *fuzzy multisets*¹⁰ based on multisets and where the membership degree of each element is given by a subset of $[0, 1]$, and so forth.

Despite the previous extensions overcome in different ways, the managing of simultaneous sources of vagueness, Torra¹¹ introduced a new extension of fuzzy sets so-called hesitant fuzzy sets (HFSs), motivated for the common difficulty that often appears when the membership degree of an element must be established and the difficulty is not because of an error margin (as in IFS) or due to some possibility distribution (as in T2FS), but rather because there are some possible values that make to hesitate about which one would be the right one. This situation is very usual in decision making when an expert might consider different degrees of membership $\{0.67, 0.72, 0.74\}$ of the element x in the set A .

HFSs have attracted the attention of many researchers in a short period of time because hesitant situations are very common in different real-world problems, and this new approach facilitates the management of uncertainty provoked by hesitation. A deep revision of the specialized literature shows the quick growth and applicability of HFSs, which have been extended from different points of view, quantitative¹²⁻¹⁵ and qualitative¹⁶ (because hesitation can arise modeling the uncertainty in both ways). In addition, many operators for HFSs and their extensions have been introduced to deal with such a type of information in different applications where decision making has been the most remarkable one.

The aim of this paper is to develop an extensive and intensive overview about HFSs paying attention not only to theoretical concepts including extensions, computational tools, or applications in which HFSs have provided satisfactory results but also fixing a consistent notation.

The paper is organized as follows: Section 2 introduces the concept of HFS, some basic operators, and their properties. Section 3 revises extensions of HFSs. Sections 4 and 5 present different aggregation operators and measures for HFSs, respectively. In Section 6, a deep review of applications which use hesitant information is presented. Section 7 points out the trends and directions of the hesitant context, and finally the paper is concluded in Section 8.

2. HESITANT FUZZY SETS: CONCEPTS, BASIC OPERATIONS, AND PROPERTIES

HFS is a novel and recent extension of fuzzy sets that aims to model the uncertainty originated by the hesitation that might arise in the assignment of membership degrees of the elements to a fuzzy set.

To show and understand the impact and usefulness of HFSs together with their extensions, operators, and applications, in this section different concepts about HFSs are revised, including basic operations and their properties. An important effort has been done to clarify and make consistent the notation about such concepts that should be used from now on to avoid misunderstandings and mistakes regarding HFSs.

2.1 Concepts

A HFS is defined in terms of a function that returns a set of membership values for each element in the domain.

DEFINITION 1¹¹. *Let X be a reference set, a HFS on X is a function \mathfrak{h} that returns a subset of values in $[0,1]$:*

$$\mathfrak{h} : X \rightarrow \wp([0, 1]) \tag{1}$$

A HFS can also be constructed from a set of fuzzy sets.

DEFINITION 2¹¹. *Let $M = \{\mu_1, \dots, \mu_n\}$ be a set of n membership functions. The HFS associated with M , \mathfrak{h}_M , is defined as*

$$\begin{aligned} \mathfrak{h}_M : X &\rightarrow \wp([0, 1]) \\ \mathfrak{h}_M(x) &= \bigcup_{\mu \in M} \{\mu(x)\} \end{aligned} \tag{2}$$

where $x \in X$.

It is remarkable that this definition is quite suitable to decision making, when experts have to assess a set of alternatives. In such a case, M represents the assessments of the experts for each alternative and \mathfrak{h}_M the assessments of the set of experts. However, note that it only allows to recover those HFSs whose memberships are given by sets of cardinality less than or equal to n .

Afterward, Xia and Xu¹⁷ completed the original definition of HFS by including the mathematical representation of a HFS as follows:

$$E = \{(x, h_E(x)) : x \in X\},$$

where $h_E(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set E . For convenience, Xia and Xu noted

$h = h_E(x)$ and called it hesitant fuzzy element (HFE) of E and $H = \cup h_E(x)$, the set of all HFEs of E .

In some papers, the concepts HFS and HFE are used indistinctively,¹³ even though both concepts are different. A HFS is a set of subsets in the interval $[0,1]$, one set for each element of the reference set X . A HFE is one of such sets, the one for a particular $x \in X$.

Given a function ϕ on n HFEs, the following definition expresses how to build a function ϕ' on HFS from ϕ :

DEFINITION 3. Let $\{H_1, \dots, H_n\}$ be a set of n HFSs on X and ϕ an n -ary function on HFEs, we define

$$\phi'(H_1, \dots, H_n)(x) = \phi(H_1(x), \dots, H_n(x)). \quad (3)$$

However, we can define functions for HFSs that do not correspond to functions on HFEs. The following definition illustrates this case:

DEFINITION 4. Let $\{H_1, \dots, H_n\}$ be a set of n HFSs on X ,

$$\phi'(H_1, \dots, H_n)(x) = \frac{(\max_i \max(H_i) + \min_i \min(H_i))}{2} \wedge \cup_i H_i(x) \quad (4)$$

where for any $\alpha \in [0, 1]$, $\alpha \wedge h$ corresponds to $\{s | s \in h, s \leq \alpha\}$ (this corresponds to h_α^- using the notation in Ref. 11).

More recently, Bedregal et al.¹⁸ have presented a particular case of HFS so-called a typical hesitant fuzzy set (THFS), which considers some restrictions.

DEFINITION 5¹⁸. Let $\mathbb{H} \subseteq \wp([0, 1])$ be the set of all finite nonempty subsets of the interval $[0,1]$, and let X be a nonempty set. A THFS A over X is given by Equation 1 where $h_E : X \rightarrow \mathbb{H}$.

Each $h_E(x) \in \mathbb{H}$ is called a typical hesitant fuzzy element of \mathbb{H} (THFE).

Remark 1. To use HFS properly, it is recommended to consider finite and nonempty HFS, that is, THFS. In this paper, we consider finite and nonempty HFS although we will keep the original names, HFS and HFE.

2.2 Basic Operations

In the seminal paper of HFSs,¹¹ Torra introduced initially several basic operations to deal with HFEs, although originally were not called HFEs. These definitions follow the approach of Definition 3, that is, a function for HFSs defined in terms of a function for HFEs.

DEFINITION 6¹¹. Given a HFE, h , its lower and upper bounds are

$$h^- = \min\{\gamma | \gamma \in h\} \quad (5)$$

$$h^+ = \max\{\gamma | \gamma \in h\} \quad (6)$$

DEFINITION 7¹¹. Let h be a HFE, its complement is defined as

$$h^c = \bigcup_{\gamma \in h} \{1 - \gamma\} \tag{7}$$

DEFINITION 8¹¹. Let h_1 and h_2 be two HFEs, their union is defined as

$$h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\} \tag{8}$$

DEFINITION 9¹¹. Let h_1 and h_2 be two HFEs, their intersection is defined as

$$h_1 \cap h_2 = \bigcap_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\} \tag{9}$$

Torra also discussed the relation between Atanassov’s IFS and HFS and proved that the envelope of a HFS constructed from the envelope of a HFE is an Atanassov’s IFS.

DEFINITION 10⁶. Let X be a reference set, an Atanassov’s IFS A on X is defined by

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\} \tag{10}$$

where the values $\mu_A(x)$ and $\nu_A(x)$, belonging to $[0, 1]$, represent the membership degree and nonmembership degree of the element x to the set A , respectively, with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$.

The envelope of a HFE is represented in the following definition:

DEFINITION 11¹¹. Let h be a HFE, the Atanassov’s IFS $A_{env}(h)$, is defined as the envelope of h , where $A_{env}(h)$ can be represented as

$$A_{env}(h) = \{(h^-, 1 - h^+)\} \tag{11}$$

A possible application of HFEs is in decision-making problems defined in quantitative contexts, where experts might hesitate among different values to assess alternatives or criteria. Thus, it is necessary to define aggregation operators to aggregate HFEs. For this reason, Torra and Narukawa¹⁹ proposed an extension principle that extends functions to HFS. This extension allows exporting operations on fuzzy sets to HFSs.

DEFINITION 12¹⁹. Let $E = \{H_1, \dots, H_n\}$ be a set of n HFSs and Θ a function, $\Theta : [0, 1]^n \rightarrow [0, 1]$, we then export Θ on fuzzy sets to HFSs defining

$$\Theta_E = \bigcup_{\gamma \in H_1(x) \times \dots \times H_n(x)} \{\Theta(\gamma)\} \tag{12}$$

This definition also follows the approach in Definition 3

An example of the extension for the Arithmetic mean by using the previous definition is

Example 1. Let $H_1(x) = \{0.5, 0.6, 0.7\}$ and $H_2(x) = \{0.5, 0.6\}$ be two HFSs, the Arithmetic mean (AM), of $H_1(x)$ and $H_2(x)$ is defined as follows:

$$AM_{H_1(x), H_2(x)} = \bigcup_{\gamma \in H_1(x) \times \dots \times H_n(x)} \{AM(\gamma)\} = \{AM(0.5, 0.5)\} \cup \{AM(0.5, 0.6)\} \cup \{AM(0.6, 0.5)\} \cup \{AM(0.6, 0.6)\} \cup \{AM(0.7, 0.5)\} \cup \{AM(0.7, 0.6)\} = \{0.5\} \cup \{0.55\} \cup \{0.6\} \cup \{0.65\} = \{0.5, 0.55, 0.6, 0.65\}$$

Note that the properties on Θ lead to related properties on Θ_E . For example, commutativity and associativity of Θ leads to commutativity and associativity of Θ_E . Similarly, we can give a monotonicity condition for Θ_E when Θ is monotonic. This is formalized in the following definitions:

DEFINITION 13. Given two HFSs H_1 and H_2 on X of the same cardinality, we define that $H_1 \geq H_2$ if $H_1(x) \geq H_2(x)$ for all x . Note that $H_1(x)$ and $H_2(x)$ are HFEs. Here, $h_1 \geq h_2$ for HFEs h_1 and h_2 if $h_1^{\sigma(j)} \geq h_2^{\sigma(j)}$ for all $j = \{1, \dots, |H_1|\}$ where $h^{\sigma(j)}$ is the j th element in h when they are ordered in a decreasing order.

DEFINITION 14. Let ϕ be a function on HFSs such that the cardinality of ϕ is the same for all HFSs. We then say that ϕ is monotonic when $\phi(E) \geq \phi(E')$ for all $E = \{H_1, \dots, H_n\}$ and $E' = \{H'_1, \dots, H'_n\}$ such that $H'_i \geq H_i$ for all $i = \{1, \dots, n\}$.

PROPOSITION 1. Let $E = \{H_1, \dots, H_n\}$ and $E' = \{H'_1, \dots, H'_n\}$ such that $H'_i \geq H_i$ for all $i = \{1, \dots, n\}$. Then, if Θ is a monotonic function, Θ_E is monotonic.

To establish an order between HFEs, Xia and Xu introduced a comparison law by defining a score function, which is defined under the following assumptions:

- The values of all the HFEs are arranged in an increasing order.
- The HFEs have the same length when they are compared. Therefore, if any two HFEs have different lengths, the shorter one will be extended by adding the maximum element until both HFEs have the same length.

DEFINITION 15¹⁷. Let h be a HFE, the score function of h is defined as follows:

$$s(h) = \frac{1}{l(h)} \sum_{\gamma \in h} \gamma, \quad (13)$$

where $l(h)$ is the number of elements in h .

Let h_1 and h_2 be two HFEs, then

if $s(h_1) > s(h_2)$, then $h_1 > h_2$;

if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Nevertheless, Farhadinia²⁰ pointed out that such a function cannot discriminate some HFEs although they are apparently different. Let see the following example:

Example 2. Let $h_1 = \{0.2, 0.3, 0.7\}$ and $h_2 = \{0.1, 0.4, 0.7\}$ be two HFEs, the results obtained using the score function are the same in spite of h_1 and h_2 are different.

$$s(h_1) = \frac{1}{3}(0.2 + 0.3 + 0.7) = 0.4$$

$$s(h_2) = \frac{1}{3}(0.1 + 0.4 + 0.7) = 0.4$$

Therefore, it was introduced a new score function for HFEs.

DEFINITION 16²⁰. Let $h = \bigcup_{\gamma \in h} \{\gamma\} = \{\gamma_j\}_{j=1}^{l(h)}$ be a HFE, the score function S of h is defined by

$$S(h) = \frac{\sum_{j=1}^{l(h)} \delta(j)\gamma_j}{\sum_{j=1}^{l(h)} \delta(j)} \tag{14}$$

where $\{\delta(j)\}_{j=1}^{l(h)}$ is a positive-valued monotonic increasing sequence of index j .

Example 3. Following the previous example, if we use the new score function, the results are

$$S(h_1) = \frac{1 * 0.2 + 2 * 0.3 + 3 * 0.7}{1 + 2 + 3} = 0.48$$

$$S(h_2) = \frac{1 * 0.1 + 2 * 0.4 + 3 * 0.7}{1 + 2 + 3} = 0.5$$

Remark 2. The new score function defined by Farhadinia allows comparing HFEs that cannot be compared by the score function introduced by Xia and Xu. However, it does not solve the problem (see the following counterexample).

Example 4. Let $h_1 = \{0.2, 0.5\}$ and $h_2 = \{0, 0.6\}$ be two HFEs, the results of applying the new score function are the following ones:

$$S(h_1) = \frac{1 * 0.2 + 2 * 0.5}{1 + 2} = 0.4$$

$$S(h_2) = \frac{1 * 0 + 2 * 0.6}{1 + 2} = 0.4$$

The result obtained is the same, despite h_1 and h_2 are different.

Note that Definition 15 is an arithmetic mean of the HFEs (i.e., $s(h)$ is the arithmetic mean of values in h) and Definition 16 is another aggregation operator of the HFEs with weights $\delta(j) / \sum \delta(j)$. Other aggregation operators (see, e.g., Ref. 21) can be used for the same purpose.

2.3 Properties

To conclude this section, some relevant properties of HFEs are reviewed.

PROPOSITION 2¹¹. *Let h be a HFE and $A_{env}(h)$ its envelope, then*

$$A_{env}(h^c) = (A_{env}(h))^c$$

Proof. \subseteq

$$A_{env}(h^c) = \langle \min(1-h), 1-\max(1-h) \rangle = \langle 1-\max h, 1-1+\min h \rangle = \langle 1-h^+, h^- \rangle$$

$$\supseteq \\ (A_{env}(h))^c = \langle h^-, 1-h^+ \rangle^c = \langle 1-h^+, h^- \rangle$$

□

PROPOSITION 3¹¹. *Let h_1 and h_2 be two HFEs, then*

$$A_{env}(h_1 \cup h_2) = A_{env}(h_1) \cup A_{env}(h_2)$$

Proof. \subseteq

$$A_{env}(h_1 \cup h_2) = \langle \max(h_1^-, h_2^-), \min(1-h_1^+, 1-h_2^+) \rangle$$

$$\supseteq \\ A_{env}(h_1) \cup A_{env}(h_2) = \langle h_1^-, 1-h_1^+ \rangle \cup \langle h_2^-, 1-h_2^+ \rangle = \\ \langle \max(h_1^-, h_2^-), \min(1-h_1^+, 1-h_2^+) \rangle.$$

□

PROPOSITION 4¹¹. *Let h_1 and h_2 be two HFEs, then*

$$A_{env}(h_1 \cap h_2) = A_{env}(h_1) \cap A_{env}(h_2)$$

The proof of the intersection is similar to the union.

PROPOSITION 5¹¹. *Let H_1 and H_2 be two HFSs with $H(x)$, a finite nonempty convex set for all $x \in X$, i.e. H_1 and H_2 are intuitionistic fuzzy sets, then,*

- H_1^c is equivalent to the complement of Atanassov's IFS,
- $H_1 \cup H_2$ is equivalent to the union of Atanassov's IFS,
- $H_1 \cap H_2$ is equivalent to the intersection of Atanassov's IFS.

The previous propositions prove that the operations defined for HFEs are consistent with the ones defined for Atanassov's IFSs. The relationship between HFEs and fuzzy multisets was also considered in Ref. 11.

3. EXTENSIONS OF HESITANT FUZZY SETS

The concepts, basic operations, and properties defined for HFSs and HFEs cope with the hesitation of assigning a membership degree of an element to a fuzzy set. The idea of modeling such a hesitation has been extended together with the previous

methods and tools to the following three situations: (i) to model the hesitation not only for the assignment of the membership degree but also for the nonmembership degree; (ii) to manage the hesitation on membership degrees that are not exactly defined but expressed by interval values, intuitionistic fuzzy sets, or triangular fuzzy numbers; (iii) to deal with the hesitation in qualitative settings in which information is linguistically modeled.

These extensions are further detailed below.

3.1 Dual Hesitant Fuzzy Sets

As we aforementioned, Atanassov’s IFS⁶ uses two functions to handle the membership and nonmembership degrees. Zhu et al. proposed the concept of dual hesitant fuzzy set (DHFS),¹⁵ as an extension of HFS to deal with the hesitation both for the membership degree and nonmembership degree.

A DHFS is defined in terms of two functions that return two sets of membership and nonmembership values, respectively, for each element in the domain as follows:

DEFINITION 17¹⁵. *Let X be a set, a DHFS D on X is defined as*

$$D = \{ \langle x, h(x), g(x) \rangle \mid x \in X \} \tag{15}$$

where $h(x)$ and $g(x)$ are two sets of values in the interval $[0, 1]$, denoting the possible membership and nonmembership degrees of the element $x \in X$ to the set D , respectively, with the following conditions:

$$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1$$

where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \max_{\gamma \in h(x)} \{ \gamma \}$, and $\eta^+ = \max_{\eta \in g(x)} \{ \eta \} \forall x \in X$.

For convenience, the pair $d(x) = (h(x), g(x))$ is called dual hesitant fuzzy element (DHFE) and noted by $d = (h, g)$.

Example 5. Let $X = \{x_1, x_2\}$ be a reference set, then D defined by

$$D = \{ \langle x_1, \{0.3, 0.4\}, \{0.5\} \rangle, \langle x_2, \{0.2, 0.4\}, \{0.2, 0.5\} \rangle \}$$

is a DHFS.

Zhu et al. defined some basic operations, such as the complement of a DHFE, the union, and intersection of two DHFEs. A comparison law was also proposed to compare DHFEs. To do so, a score function and accuracy function were introduced.

Motivated by the extension principle presented by Torra and Narukawa,¹⁹ an extension principle based on the ordered modular average operator²² was proposed to develop some basic operations and aggregations operators for DHFEs.¹⁵ Furthermore, a practical example of group forecasting was shown. The example represents assessments by DHFEs.

3.2 Interval-Valued Hesitant Fuzzy Sets

In many real decision-making problems, the available information is not enough, since it might be difficult for experts to provide their preferences using crisp values. A possible solution is to represent such preferences by interval values. Therefore, keeping in mind the concept of HFS, Chen et al. presented the definition of interval-valued hesitant fuzzy set (IVHFS),¹² as a generalization of HFS in which the membership degrees of an element to a given set are defined by several possible interval values.

An IVHFS is defined as follows:

DEFINITION 18¹². Let X be a reference set, and $I([0, 1])$ be a set of all closed subintervals of $[0, 1]$. An IVHFS on X is

$$\tilde{A} = \{\langle x_i, \tilde{h}_A(x_i) \rangle | x_i \in X, i = 1, \dots, n\} \quad (16)$$

where $\tilde{h}_A(x_i) : X \rightarrow \wp(I([0, 1]))$ denotes all possible interval-valued membership degrees of the element $x_i \in X$ to the set \tilde{A} .

For convenience, $\tilde{h}_A(x_i)$ is called an interval-valued hesitant fuzzy element (IVHFE), where each $\tilde{\gamma} \in \tilde{h}_A(x_i)$ is an interval and $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$, being $\tilde{\gamma}^L$ and $\tilde{\gamma}^U$ the lower and upper limits of $\tilde{\gamma}$, respectively.

Example 6. Let $X = \{x_1, x_2\}$ be a reference set a IVHFS \tilde{A} , could be as follows:

$$\tilde{A} = \{\langle x_1, \{[0.1, 0.2], [0.3, 0.5]\} \rangle, \langle x_2, \{[0.2, 0.4], [0.5, 0.6], [0.7, 0.9]\} \rangle\}$$

Note that when the upper and lower limits of the interval values are equal, the IVHFS becomes a HFS.

Chen et al. presented some operational laws, such as the union, intersection, and complement and studied their properties. They also defined a score function to obtain an order between two IVHFEs. To calculate the distance between two IVHFEs, two distance measures that can be considered as extensions of the Hamming and Euclidean distances were proposed. A series of specific aggregation operators for IVHFEs such as, the interval-valued hesitant fuzzy weighted averaging, interval-valued hesitant fuzzy weighted geometric, interval-valued hesitant fuzzy ordered weighted averaging, interval-valued hesitant fuzzy ordered weighted geometric and their generalizations were defined in Ref. 12. In addition, a group decision-making approach based on interval-valued hesitant preference relations was developed.

3.3 Generalized Hesitant Fuzzy Sets

Sometimes, experts might hesitate among several possible membership degrees with the form of both crisp values and interval values in $[0, 1]$. To handle directly this type of assessments in a decision-making process, Qian et al. extended the concept of HFS by Atanassov's IFS. The idea consists of representing the membership as the union of some Atanassov's IFS.⁶ To define this new extension called generalized hesitant fuzzy set (GHFS), authors use Equation 2 introduced by Torra.

DEFINITION 19¹³. Given a set of n membership functions,

$$M = \{\alpha_i = (\mu_i, \nu_i) | 0 \leq \mu_i, \nu_i \leq 1, 0 \leq \mu_i + \nu_i \leq 1, i = \{1, \dots, n\}\}, \quad (17)$$

the GHFS associated with M , \mathfrak{h}_M , is defined as follows:

$$\mathfrak{h}_M(x) = \cup_{(\mu_i(x), \nu_i(x)) \in M} (\mu_i(x), \nu_i(x)). \quad (18)$$

Remark 3. Notice that a GHFS extends slightly the concept of DHFS¹⁵ as we can see in the following example:

Example 7. Let $X = \{x_1\}$ be a reference set, then

$$\mathfrak{h}_M(x_1) = \{(0.5, 0.3), (0.6, 0.3), (0.4, 0.5)\}$$

is a GHFS.

In this example, $\gamma^+ = 0.6$ and $\eta^+ = 0.5$, therefore $0.6 + 0.5 > 1$; it does not achieve the restriction to be a DHFS.

Similar to Ref. 11, it was defined the complement, union and intersection of GHFSs, as well as, the envelope of a GHFS. Some properties and relationships with HFSs were also discussed in Ref. 13. A comparison law was introduced to compare two GHFSs according to the score and consistency functions defined for this type of information.

Usually, in decision-making problems, it is necessary to use aggregation techniques to aggregate the assessments provided by experts and obtain collective values for the alternatives to select the best one. Therefore, to aggregate a set of GHFSs it was proposed an extension principle that extends the operations for Atanassov's IFSSs to GHFSs. A decision support system framework based on GHFS was developed to support the activities carried out in decision-making processes.

3.4 Triangular Fuzzy Hesitant Fuzzy Sets

Yu pointed out in Ref. 14 that sometimes it is difficult for experts to express the membership degrees of an element to a given set only by crisp values between 0 and 1. To model this hesitation, Yu introduced the concept of triangular fuzzy hesitant fuzzy set (TFHFS), whose membership degrees of an element to a fuzzy set are expressed by several triangular fuzzy numbers.²³

DEFINITION 20¹⁴. Let X be a fixed set, a TFHFS \tilde{E} on X is defined in terms of a function $\tilde{f}_{\tilde{E}}(x)$ that returns several triangular fuzzy values,

$$\tilde{E} = \{ \langle x, \tilde{f}_{\tilde{E}}(x) \rangle \mid x \in X \} \quad (19)$$

where $\tilde{f}_{\tilde{E}}(x)$ is a set of several triangular fuzzy numbers, which express the possible membership degrees of an element $x \in X$ to a set \tilde{E} . $\tilde{f}_{\tilde{E}}(x)$ is called triangular fuzzy hesitant fuzzy element (TFHFE) and noted $(\tilde{f})_{\tilde{E}}(x_i) = \{(\tilde{\xi}^L, \tilde{\xi}^M, \tilde{\xi}^U) \mid \tilde{\xi} \in \tilde{f}_{\tilde{E}}(x_i)\}$.

Example 8. Let $X = \{x_1, x_2\}$ be a reference set, \tilde{E} defined by

$$\tilde{E} = \{\langle x_1, \{(0.1, 0.3, 0.5), (0.4, 0.6, 0.8)\} \rangle, \langle x_2, \{(0.1, 0.2, 0.3)\} \rangle\}$$

is a TFHFS.

Note that if $\tilde{\xi}^L = \tilde{\xi}^M = \tilde{\xi}^U$, then the TFHFS becomes a HFS.

Some basic operations such as the addition and multiplication of TFHFEs were defined. A comparison law was proposed by means of the definition of a score and accuracy functions. Different specific aggregation operators for TFHFEs such as, triangular fuzzy hesitant fuzzy weighted averaging, triangular fuzzy hesitant fuzzy weighted geometric and their generalizations were also introduced.

Furthermore, a multicriteria decision-making model in which experts can express their assessments by using TFHFEs was presented and applied to solve a teaching quality evaluation problem.

3.5 Hesitant Fuzzy Linguistic Term Sets

The previous extensions suit problems that are defined in quantitative situations, but uncertainty is often because of the vagueness of meanings that are used by experts in problems whose nature is rather qualitative. Different linguistic models have been presented in the literature^{24–28} to model linguistic information. Nevertheless, when experts face decision situations with high degree of uncertainty, they often hesitate among different linguistic terms and would like to use more complex linguistic expressions which cannot be expressed with the classical linguistic approaches. This limitation is due to the use of linguistic terms defined a priori, and because most linguistic approaches model the information by using only one linguistic term. To overcome this limitation, different proposals have been introduced in the literature^{29–33} to provide more flexible and richer expressions, which can include more than one linguistic term. Notwithstanding, neither of them is adequate to fulfill the necessities and requirements of experts in hesitant situations. Consequently, bearing in mind the idea under the HFS,¹¹ Rodríguez et al. proposed the concept of hesitant fuzzy linguistic term set (HFLTS),¹⁶ which keeps the basis on the fuzzy linguistic approach³⁴ and extends the idea of HFS to linguistic contexts.

Formally, a HFLTS is defined as follows:

DEFINITION 21¹⁶. Let $S = \{s_0, \dots, s_g\}$ be a linguistic term set, a HFLTS H_S is defined as an ordered finite subset of consecutive linguistic terms of S :

$$H_S = \{s_i, s_{i+1}, \dots, s_j\} \text{ such that } s_k \in S, k \in \{i, \dots, j\}$$

Example 9. Let S be a linguistic term set, $S = \{s_0 : \text{nothing}, s_1 : \text{very_low}, s_2 : \text{low}, s_3 : \text{medium}, s_4 : \text{high}, s_5 : \text{very_high}, s_6 : \text{perfect}\}$, and ϑ a linguistic variable, then $H_S(\vartheta)$ defined by

$$H_S(\vartheta) = \{\text{high}, \text{very_high}, \text{perfect}\}$$

is a HFLTS.

Remark 4. The use of consecutive linguistic terms in HFLTS is because of a cognitive point of view in which in a discrete domain with a short number of terms (usually not more than nine) makes no sense to hesitate among arbitrary and total different linguistic terms, $\{low, high, very_high\}$ and not hesitate in their middle terms. The use of comparative linguistic expressions³⁵ is a clear example of hesitation among several linguistic terms by human beings. Using the HFLTS as the natural representation for managing the comparative expressions in decision making.

Rodríguez et al. introduced some basic operations for HFLTS, such as the complement, union, and intersection and studied diverse properties over them. Similarly to the concept of envelope of a HFS, it was defined the envelope of a HFLTS. This envelope was used to propose a comparison law for HFLTSs. Besides, two symbolic aggregation operators, *min_upper* and *max_lower* were developed to aggregate HFLTSs.

The concept of HFLTS was introduced as something that can be used directly by experts to elicit several linguistic terms, but such elements are not similar to human beings' way of thinking. Therefore, Rodríguez et al. proposed the use of context-free grammars to generate linguistic expressions similar to human beings' expressions that are easily represented by HFLTS. A context-free grammar G_H , which may generate comparative linguistic expressions close to the expressions used by experts in decision-making problems, was introduced in Ref. 16 and extended in Ref. 35.

DEFINITION 22³⁵. Let G_H be a context-free grammar and $S = \{s_0, \dots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows: $V_N = \{\langle primary\ term \rangle, \langle composite\ term \rangle, \langle unary\ relation \rangle, \langle binary\ relation \rangle, \langle conjunction \rangle\}$
 $V_T = \{lower\ than, greater\ than, at\ least, at\ most, between, and, s_0, s_1, \dots, s_g\}$
 $I \in V_N$
 $P = \{I ::= \langle primary\ term \rangle | \langle composite\ term \rangle$
 $\langle composite\ term \rangle ::= \langle unary\ relation \rangle \langle primary\ term \rangle | \langle binary\ relation \rangle$
 $\langle primary\ term \rangle \langle conjunction \rangle \langle primary\ term \rangle$
 $\langle primary\ term \rangle ::= s_0 | s_1 | \dots | s_g$
 $\langle unary\ relation \rangle ::= lower\ than | greater\ than | at\ least | at\ most$
 $\langle binary\ relation \rangle ::= between$
 $\langle conjunction \rangle ::= and\}$

To obtain HFLTS from the comparative linguistic expressions generated by the context-free grammar G_H , a transformation function E_{G_H} was proposed.

The use of comparative linguistic expressions based on context-free grammars and HFLTS has been applied to different decision-making problems.^{16,35-37}

Although the definition of HFLTS is very recent, it has received a lot of attention by other researchers in the community, and there are already different proposals based on this definition.³⁶⁻⁴⁰

Lee and Cheng have pointed out that the multicriteria decision making- model proposed in Ref. 16 is too complex to solve decision-making problems. Therefore, a new decision-making model based on likelihood-based comparison relation of

HFLTS has been introduced.³⁸ To do so, the concept of likelihood-based comparison relations of HFLTS and a similarity measure was introduced.

Zhu and Xu have introduced the concept of hesitant fuzzy linguistic preference relation (HFLPR),⁴⁰ which is a matrix $B = (b_{ij})_{n \times n} \subset X \times X$ where b_{ij} is a HFLTS. Owing to the importance of the consistency measures using preference relations, such authors have defined some consistency measures for the HFLPR and a consistency index that establishes the consistency thresholds of the HFLPR to measure whether a HFLPR is of acceptable consistency. In addition, two optimization methods have been developed to improve the consistency in the HFLPR when the consistency is unacceptable. Zhang et al. also proposed a discrete region-based approach to improve the consistency of the pairwise comparison matrix by using HFLTS.⁴¹

Wei et al. indicate in Ref. 39 that the comparison method for HFLTS presented in Ref. 16 might provide results that do not correspond with the common sense, because if two HFLTSs have some common linguistic terms, it does not seem reasonable to say that one HFLTS is absolutely superior to another. To overcome this shortcoming, a new comparison method for HFLTSs that uses the probability theory is introduced. These authors also presented two new linguistic aggregation operators for HFLTSs, which take into account the importance of criteria and/or expert in decision-making problems. Such operators generalize the linguistic weighted average and linguistic ordered weighted average to HFLTSs. More aggregation operators can be found in Ref. 42.

Recently, Liu and Rodríguez have pointed out³⁷ that the semantics of the comparative linguistic expressions based on a context-free grammar and HFLTSs should be represented by fuzzy membership functions instead of linguistic intervals,¹⁶ since the concept of HFLTS is based on the fuzzy linguistic approach, and the linguistic terms of the fuzzy linguistic approach are represented by a syntax and fuzzy semantics. Therefore, a new fuzzy representation for comparative linguistic expressions based on a fuzzy envelope has been introduced.³⁷

3.6 Summary on HFSs Extensions

In this section, we have reviewed existing extensions of HFS. We have seen that one of the extensions was based on considering hesitancy on a set of linguistic terms. This has opened a new line of research. Other approaches consider some degrees of fuzziness on the values in the interval $[0,1]$. The latter class can be seen as different variations of type-2 HFS. Each value is replaced by a fuzzy set (an interval or a triangular fuzzy set). Type- n HFS is a natural extension of all these approaches. Note that type- n HFS can be represented by type $(n + 1)$ fuzzy sets.

4. AGGREGATION OPERATORS FOR HFES

As we will see in Section 6, HFES have been applied in different fields being decision making the most remarkable one. A basic scheme of a decision-making problem mainly consists of two phases, *aggregation* and *exploitation* (see Figure 1).

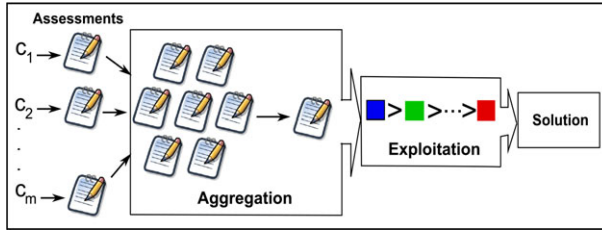


Figure 1. Basic scheme of a decision-making problem.

In the aggregation phase, the information is grouped to reflect a collective value for each alternative or criterion, and, in the exploitation phase, the best alternative is selected as a solution to the decision problem by using the collectives values obtained in the previous phase. Thus, different aggregation techniques have been developed to carry out decision-making processes in which experts express their assessments by using HFEs.

Very relevant examples of aggregation operators often used for decision making are based on the arithmetic mean, geometric mean, and integrals. Many generalizations and extensions of these operators have been proposed thereafter to aggregate different types of information, such as IVFS, Atanassov’s IFS, etc. We will focus on the aggregation operators defined for HFEs.

Xia and Xu presented in¹⁷ two main aggregation operators, such as hesitant fuzzy weighted averaging and hesitant fuzzy weighted geometric, which are defined as follows:

DEFINITION 23¹⁷. Let h_i ($i = 1, \dots, n$) be a collection of HFEs, $h_i \in H$, the hesitant fuzzy weighted averaging (HFWA) operator is a mapping $H^n \rightarrow H$ such that

$$HFWA(h_1, \dots, h_n) = \bigoplus_{i=1}^n (w_i h_i) = \cup_{\gamma_1 \in h_1, \dots, \gamma_n \in h_n} \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i)^{w_i} \right\}. \quad (20)$$

where $w = (w_1, \dots, w_n)^T$ is a weighting vector with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. In case of $w = (1/n, \dots, 1/n)^T$, then the HFWA operator reduces to the hesitant fuzzy averaging (HFA) operator:

$$HFA(h_1, \dots, h_n) = \bigoplus_{i=1}^n \left(\frac{1}{n} h_i \right) = \cup_{\gamma_1 \in h_1, \dots, \gamma_n \in h_n} \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i)^{1/n} \right\}. \quad (21)$$

In this definition, $\bigoplus_{i=1}^n (w_i h_i)$ is a shortcut of the term of the right of Equation 20.

DEFINITION 24¹⁷. Let h_i ($i = 1, \dots, n$) be a collection of HFEs, the hesitant fuzzy weighted geometric (HFWG) operator is a mapping $H^n \rightarrow H$ such that

$$HFWG(h_1, \dots, h_n) = \bigotimes_{i=1}^n h_i^{w_i} = \cup_{\gamma_1 \in h_1, \dots, \gamma_n \in h_n} \left\{ \prod_{i=1}^n \gamma_i^{w_i} \right\}. \quad (22)$$

where $w = (w_1, \dots, w_n)^T$ is the weighting vector with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. In the case of $w = (1/n, \dots, 1/n)^T$, then the HFWG operator reduces to the hesitant fuzzy geometric (HFG) operator:

$$HFG(h_1, \dots, h_n) = \bigotimes_{i=1}^n h_i^{1/n} = \cup_{\gamma_1 \in h_1, \dots, \gamma_n \in h_n} \left\{ \prod_{i=1}^n \gamma_i^{1/n} \right\}. \quad (23)$$

In this definition, $\bigotimes_{i=1}^n h_i^{w_i}$ is a shortcut of the term of the right in Equation 22.

Different extensions and generalizations of these operators, such as generalized hesitant fuzzy weighted averaging, generalized hesitant fuzzy weighted geometric, hesitant fuzzy ordered weighted averaging, hesitant fuzzy ordered weighted geometric, generalized hesitant fuzzy ordered weighted averaging, generalized hesitant fuzzy ordered weighted geometric, hesitant fuzzy hybrid averaging, hesitant fuzzy hybrid geometric, generalized hesitant fuzzy hybrid averaging, and Generalized hesitant fuzzy hybrid geometric were presented in Ref. 17.

Yu et al proposed in Ref. 43 a new hesitant fuzzy aggregation operator based on the Choquet integral whose fundamental feature is that it does not only consider the importance of the elements or their ordered positions but also takes into account a fuzzy measure used to express the relationship between the information sources (e.g., experts) that supplied h_i .

DEFINITION 25⁴³. Let ρ be a fuzzy measure on X , and h_i ($i = 1, 2, \dots, n$) be a set of HFEs, the hesitant fuzzy Choquet integral (HFCI) operator is defined as follows.

$$\int h_i d\rho = HFCI(h_1, \dots, h_n) = \bigotimes_{i=1}^n h_{\sigma(i)}^{\rho(B_{\sigma(i)}) - \rho(B_{\sigma(i-1)})} \quad (24)$$

where $(\sigma(1), \dots, \sigma(n))$ is a permutation of $(1, \dots, n)$, such that $h_{\sigma(1)} \geq h_{\sigma(2)} \geq \dots, \geq h_{\sigma(n)}$, $B_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$, for $k \geq 1$, and $B_{\sigma(0)} = \emptyset$.

Remark 5. To use the HFCI operator, it is necessary to fix a lineal order on the HFEs.

Motivated by the idea of prioritized aggregation operators, Wei proposed the hesitant fuzzy prioritized weighted average and hesitant fuzzy prioritized weighted geometric aggregation operators,⁴⁴ which take into account different priority levels for the criteria defined in a multicriteria decision-making problem.

DEFINITION 26⁴⁴. Let $h_i (i = 1, \dots, n)$ be a set of HFEs, the hesitant fuzzy prioritized weighted average (HFPWA) operator is defined as follows:

$$HFPWA(h_1, \dots, h_n) = \bigoplus_{i=1}^n \left(\frac{T_i h_i}{\sum_{i=1}^n T_i} \right) \tag{25}$$

where $T_i = \prod_{k=1}^{i-1} s(h_k) (i = 2, \dots, n)$, $T_1 = 1$ and $s(h_k)$ is the score values of $h_k (i = 1, \dots, n)$.

Based on the HFPWA operator and the geometric mean, it was defined the hesitant fuzzy prioritized weighted geometric (HFPWG) operator.

DEFINITION 27⁴⁴. Let $h_i (i = 1, \dots, n)$ be a set of HFEs, the HFPWG operator is

$$HFPWG(h_1, \dots, h_n) = \bigotimes_{i=1}^n h_i^{\frac{T_i}{\sum_{i=1}^n T_i}} \tag{26}$$

where $T_i = \prod_{k=1}^{i-1} s(h_k) (i = 2, \dots, n)$, $T_1 = 1$, and $s(h_k)$ is the score values of $h_k (i = 1, \dots, n)$.

The generalizations of such operators have been proposed by Yu et al.⁴⁵

The Bonferroni mean is an aggregation operator very useful in some applications because it captures the interrelationship between the input arguments. Yu and Zhou extended this operator to hesitant fuzzy environment and defined the generalized hesitant fuzzy Bonferroni mean (GHFBM).

DEFINITION 28⁴⁶. Let $p, q, r > 0$, and $h_i (i = 1, \dots, n)$ be a collection of HFEs on X , the GHFBM is defined as follows:

$$GHFBM^{p,q,r}(h_1, \dots, h_n) = \left(\frac{1}{n(n-1)(n-2)} \left(\bigoplus_{i,j,k=1, i \neq j \neq k}^n ((h_i)^p \otimes (h_j)^q \otimes (h_k)^r) \right) \right)^{\frac{1}{p+q+r}} \tag{27}$$

Other extensions of Bonferroni mean such as hesitant fuzzy geometric Bonferroni mean, hesitant fuzzy Choquet geometric Bonferroni mean, weighted hesitant fuzzy geometric Bonferroni mean, weighted hesitant fuzzy Choquet geometric Bonferroni mean and weighted hesitant fuzzy Bonferroni mean were proposed in Refs. 47 and 48.

Another family of aggregation operators is the quasi-arithmetic means.⁴⁹ An extension of such operators to aggregate HFEs has been introduced in Ref. 50.

DEFINITION 29⁵⁰. Let $h_i (i = 1, \dots, n)$ be a set of HFEs and $w = (w_1, \dots, w_n)^T$ the weighting vector, such that $\sum_{i=1}^n w_i = 1$ and $w_i \geq 0, i = 1, \dots, n$. The

quasi-hesitant fuzzy weighted average (QHFWA) aggregation operator is defined as

$$QHFWA(h_1, \dots, h_n) = \cup_{\gamma_{\sigma(i)} \in h_{\sigma(i)}, i=1, \dots, n} \left\{ g^{-1} \left(\sum_{i=1}^n w_i g(\gamma_i) \right) \right\} \quad (28)$$

where $g(\gamma)$ is a continuous strictly monotonic function.

By using the ordered modular average (OMA) proposed by Mesiar and Mesiarova-Zemankova,²² the QHFWA was generalized as follows:

DEFINITION 30⁵⁰. Let $h_i (i = 1, \dots, n)$ be a set of HFEs and $w = (w_1, \dots, w_n)^T$ the weighting vector, such that $\sum_{i=1}^n w_i = 1$ and $w_i \geq 0, i = 1, \dots, n$. The hesitant fuzzy modular weighted average (HFMTWA) operator is

$$HFMTWA(h_1, \dots, h_n) = \cup_{\gamma_i \in h_i, i=1, \dots, n} \left\{ \sum_{i=1}^n w_i f_i(\gamma_i) \right\} \quad (29)$$

where $f_i (i = 1, \dots, n)$ are strictly continuous monotonic functions.

More extensions of these operators can be found in Ref. 50.

A variety of hesitant fuzzy power aggregation operators and their relationships are introduced in Refs. 51 and 52.

DEFINITION 31⁵². Let $h_i (i = 1, \dots, n)$ be a set of HFEs, the hesitant fuzzy power average (HFPA) operator is defined as follows:

$$HFPA(h_1, \dots, h_n) = \frac{\oplus_{i=1}^n (1 + T(h_i)) h_i}{\sum_{i=1}^n (1 + T(h_i))} \quad (30)$$

where $T(h_i) = \sum_{i=1, i \neq j}^n \text{Sup}(h_i, h_j)$ and $\text{Sup}(h_i, h_j)$ are the support for h_i from h_j that satisfies the following properties:

- $\text{Sup}(h_i, h_j) \in [0, 1]$,
- $\text{Sup}(h_i, h_j) = \text{Sup}(h_j, h_i)$, and
- $\text{Sup}(h_i, h_j) \geq \text{Sup}(h_s, h_t)$, if $d(h_i, h_j) < d(h_s, h_t)$. Being d a distance measure between two HFEs.

The support (i.e., Sup) is essentially a similarity measure that can be used to measure the proximity of HFEs. The higher similarity, the smaller distance between two HFEs, and, therefore, more support each other.

DEFINITION 32⁵². Let $h_i (i = 1, \dots, n)$ be a set of HFEs, the hesitant fuzzy power geometric (HFPG) operator is defined as follows:

$$HFPG(h_1, \dots, h_n) = \frac{\otimes_{i=1}^n (1 + T(h_i)) h_i}{\sum_{i=1}^n (1 + T(h_i))} \quad (31)$$

where $T(h_i) = \sum_{i=1, i \neq j}^n Sup(h_i, h_j)$ and $Sup(h_i, h_j)$ is the support for h_i from h_j .

Diverse extensions and generalizations of these two operators are also presented in Refs. 51 and 52.

More recently, Bedregal et al.¹⁸ have proposed two methodologies to produce triangular hesitant aggregation functions over all THFS, i.e., for all finite nonempty subsets of the unitary interval [0,1] (see Definition 5). It has also been introduced the finite hesitant triangular norms, studying their main properties and analyzing the action of H-automorphisms over such operators.

Note that most of these definitions can be seen as applications of the extension principle (Definition 12) with an appropriate function Θ . For example, Definition 29 corresponds to the extension principle when Θ is the quasiweighted mean. Note also that the properties of the function Θ will be inherited by Θ_E as discussed in Section 2.

Remark 6. Aggregation operators are a type of monotonic functions, so they are related to a partial order. However, there is not a linear order for HFS.

5. DISTANCE MEASURES, CORRELATION COEFFICIENTS, AND INFORMATION MEASURES

This section reviews diverse measures for hesitant fuzzy environment such as distance and similarity measures, correlation coefficients, and information measures.

5.1 Distance Measures

Distance and similarity measures are important tools for distinguishing the difference between two objects and have become important due to the significant applications in diverse fields, such as machine learning, pattern recognition, and decision making. Several of them are the Hamming, Euclidean, and Hausdorff distances. These measures have been extended to manage different types of information.

Xu and Xia proposed several distance and similarity measures that satisfy the following properties.

DEFINITION 33⁵³. *Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, then the distance measure between M and N is defined as $d(M, N)$ and satisfies the following properties:*

1. $0 \leq d(M, N) \leq 1$;
2. $d(M, N) = 0$ iff $M = N$;
3. $d(M, N) = d(N, M)$.

DEFINITION 34⁵³. *Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, then the similarity measure between M and N is defined as $s(M, N)$ and satisfies the following properties:*

1. $0 \leq s(M, N) \leq 1$;
2. $s(M, N) = 1$ iff $M = N$;
3. $s(M, N) = s(N, M)$.

Based on the Hamming and Euclidean distances, the hesitant normalized Hamming distance and hesitant normalized Euclidean distance were defined.

DEFINITION 35⁵³. Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, the hesitant normalized Hamming distance is defined as follows:

$$d_{hnh}(M, N) = \frac{1}{n} \sum_{i=1}^n \left[\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)| \right] \quad (32)$$

where $h_M^{\sigma(j)}(x_i)$ and $h_N^{\sigma(j)}(x_i)$ are the j th largest values in $h_M(x_i)$ and $h_N(x_i)$, respectively, and $l_{x_i} = \max\{l(h_M(x_i)), l(h_N(x_i))\}$ for each $x_i \in X$ being $l(h_M(x_i))$, $l(h_N(x_i))$ is the number of values of $h_M(x_i)$ and $h_N(x_i)$.

Remark 7. In many cases, the number of values of two HFEs is different, $l(h_M(x_i)) \neq l(h_N(x_i))$; therefore, to carry out the computations correctly, it is necessary that the two HFEs have the same length when they are compared. To do so, Xu and Xia fixed the following rule:

If the length between two HFEs is different, then the shorter one should be extended by adding the same value several times until both of them have the same length. The selection of such a value depends on two points of view: optimistic and pessimistic. From an optimistic point of view the value added is the maximum value of the HFE, whereas from a pessimistic point of view the value is the minimum one of the HFE.

DEFINITION 36⁵³. Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, the hesitant normalized Euclidean distance is defined as follows:

$$d_{hne}(M, N) = \left[\frac{1}{n} \sum_{i=1}^n \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|^2 \right) \right]^{1/2} \quad (33)$$

Equations 32 and 33 can be extended into a generalized hesitant normalized distance.

DEFINITION 37⁵³. Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, the hesitant normalized Euclidean distance is defined as follows:

$$d_{hne}(M, N) = \left[\frac{1}{n} \sum_{i=1}^n \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|^\lambda \right) \right]^{1/\lambda} \quad (34)$$

where $\lambda > 0$.

And by applying the Hausdorff distance to the previous measure, generalized hesitant normalized Hausdorff distance is obtained.

DEFINITION 38⁵³. Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, the generalized hesitant normalized Hausdorff distance is defined as follows:

$$d_{ghnh}(M, N) = \left[\frac{1}{n} \sum_{i=1}^n \max_j |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|^\lambda \right]^{1/\lambda} \tag{35}$$

with $\lambda > 0$.

Different extensions and generalizations from such definitions were also developed as well as studied their properties and relations among them.⁵³

The corresponding similarity measures for HFSs can be obtained by means of the relationship between distance and similarity measure, $s(M, N) = 1 - d(M, N)$.

Based on the distance measures presented in Ref. 53, Xu and Xia proposed some distance measures for HFEs and discussed some of their properties.⁵⁴

Zhou and Li pointed out in Ref. 55 that the distance and similarity measures presented by Xu and Xia⁵³ only satisfy three properties, and they should fulfill four properties such as the notions of fuzzy sets,⁵⁶ Atanassov's IFS,⁵⁷ IVFS,⁵⁸ and T2FS.⁵⁹ Therefore, Zhou and Li modified the axiom definitions of distance and similarity measures for HFSs by adding a new property to Definitions 33 and 34.

For the distance measures,

- Let R be a HFS, if $M \subseteq N \subseteq R$, then, $d(M, N) \leq d(M, R)$ and $d(N, R) \leq d(M, R)$;

For the similarity measures,

- Let R be a HFS, if $M \subseteq N \subseteq R$, then, $s(M, R) \leq s(M, N)$ and $s(M, R) \leq s(N, R)$;

Zhou and Li proposed some new distance and similarity measures between HFSs based on the Hamming distance, the Euclidean distance, and L_p metric and exponential operations. We are focusing on the last two because the other are quite similar to those proposed by Xu and Xia.⁵³

If the L_p metric is applied to the distance measure between HFSs, the hesitant L_p distance is defined as follows:

DEFINITION 39⁵⁵. Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, the hesitant L_p distance is

$$d_{hl_p}(M, N) = \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^{l_{x_i}} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|^p \right)^{1/p} \tag{36}$$

with $p \geq 1$.

DEFINITION 40⁵⁵. Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, an exponential-type distance measure is defined as follows:

$$d_{exp}(M, N) = \frac{1 - \exp(-d(M, N))}{1 - \exp(-d_{max})} \quad (37)$$

with d a distance measure^{53,55} and $d_{max} = \max\{d(M, N)\}$.

The similarity measures can be obtained from the distance measures.⁵⁵

In Ref. 60, it is pointed out that the generalized hesitant fuzzy weighted distance (GHFWD) and generalized hesitant fuzzy ordered weighted distance (GHFOWD) measures introduced by Xu and Xia⁵³ have the following drawbacks:

- The GHFWD measure only focuses on the weight of individual distance for each value of the HFS, but ignores the position weight with respect to the individual distance.
- The GHFOWD measure only takes into account the position weight with respect to the individual distance and does not consider the weight of the individual distance.

Therefore, GHFWD and GHFOWD measures consider only one type of importance. To overcome this limitation, Peng et al. presented a novel generalized hesitant fuzzy synergetic weighted distance measure (GHFSWD), which reflects both the individual distances and their ordered positions.

DEFINITION 41⁶⁰. Let M and N be two HFSs on $X = \{x_1, \dots, x_n\}$, then a GHFSWD measure of M and N is a mapping $GHFSWD : H^n \times H^n \rightarrow [0, 1]$, which has associated a weighted vector $w = \{w_1, \dots, w_n\}^T$, with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, such that,

$$GHFSWD(M, N) = \left(\frac{\sum_{i=1}^n \omega_i \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|^\lambda \right) w_{\rho(i)}}{\sum_{i=1}^n \omega_i w_{\rho(i)}} \right)^{1/\lambda} \quad (38)$$

where $\lambda > 0$, $\rho : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is a permutation function such that $(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|)$ is the $\rho(i)$ th largest element of the collection of individual distances $(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|)$ ($i = 1, \dots, n$), and $\omega = (\omega_1, \dots, \omega_n)^T$ is the corresponding weighting vector of the $(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|)$ ($i = 1, \dots, n$), with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$ (to compute the weighting vector see⁶¹).

Different properties of the GHFSWD measure have been studied in detail in Ref. 60.

5.2 Correlation Coefficients

The correlation reflects a linear relationship between two variables. It is an important measure in data analysis; therefore, diverse correlation coefficients have been defined for different types of information.⁶²

Chen et al.⁶³ defined the informational energy for HFSs and the corresponding correlation between HFSs.

DEFINITION 42⁶³. Let M be a HFS on $X = \{x_1, \dots, x_n\}$, the informational energy of the HFS M is defined as follows:

$$E_{HFS}(M) = \sum_{i=1}^n \left(\frac{1}{l(h_M(x_i))} \sum_{j=1}^{l_{x_i}} h_{M\sigma(j)}^2(x_i) \right) \tag{39}$$

where $h_{M\sigma(j)}(x_i)$ are the j th largest values of $h_M(x_i)$ and $l(h_M(x_i))$ is the number of values in $h_M(x_i)$.

Remark 8. In this equation, the ordering of the elements is not relevant because the sum does change.

DEFINITION 43⁶³. Let M and N be two HFS on $X = \{x_1, \dots, x_n\}$; the correlation between M and N is defined by

$$C_{HFS}(M, N) = \sum_{i=1}^n \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} h_{M\sigma(j)}(x_i) h_{N\sigma(j)}(x_i) \right) \tag{40}$$

where $l_{x_i} = \max\{l(h_M(x_i)), l(h_N(x_i))\}$.

Let M, N be two HFSs, the correlation satisfies:

- $C_{HFS}(M, M) = E_{HFS}(M)$ and
- $C_{HFS}(M, N) = C_{HFS}(N, M)$.

By using Definitions 42 and 43, the following correlation coefficient is obtained:

DEFINITION 44⁶³. Let M and N be two HFS on $X = \{x_1, \dots, x_n\}$; the correlation coefficient between M and N is

$$CC_{HFS}(M, N) = \frac{C_{HFS}(M, N)}{[C_{HFS}(M, M)]^{1/2}[C_{HFS}(N, N)]^{1/2}}$$

$$= \frac{\sum_{i=1}^n \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} h_{M\sigma(j)}(x_i) h_{N\sigma(j)}(x_i) \right)}{\left[\sum_{i=1}^n \frac{1}{l(h_M(x_i))} \sum_{j=1}^{l_{x_i}} h_{M\sigma(j)}^2(x_i) \right]^{1/2} \left[\sum_{i=1}^n \left(\frac{1}{l(h_N(x_i))} \sum_{j=1}^{l_{x_i}} h_{N\sigma(j)}^2(x_i) \right) \right]^{1/2}} \tag{41}$$

Other definitions of correlation coefficients for HFSs based on Equation 41 can be found in Ref. 63. Xu and Xia introduced in Ref. 54 the concept of the correlation coefficient for HFEs and proposed several correlation coefficient formulas.

5.3 Information Measures

Some information measures such as entropy and cross-entropy have been defined for different types of information. The entropy is the measure of fuzziness⁶⁴, and cross-entropy measures the discrimination of information. In Ref. 65, the entropy and cross-entropy for HFEs were developed.

DEFINITION 45⁶⁵. Let h_1 and h_2 be two HFEs and $h_1^{\sigma(i)}$ ($i = 1, \dots, l(h_1)$) the i th smallest value in h_1 , the entropy on h_1 is a real-valued function $E : H \rightarrow [0, 1]$, which satisfies the following axioms:

1. $E(h_1) = 0$, iff $h_1 = 0$ or $h_1 = 1$.
2. $E(h_1) = 1$, iff $h_1^{\sigma(i)} + h_1^{\sigma(l(i)-i+1)} = 1$, for $i = \{1, \dots, l\}$.
3. $E(h_1) \leq E(h_2)$, if $h_1^{\sigma(i)} \leq h_2^{\sigma(i)}$ for $h_2^{\sigma(i)} + h_2^{\sigma(l(i)-i+1)} \leq 1$ or $h_1^{\sigma(i)} \geq h_2^{\sigma(i)}$, for $h_2^{\sigma(i)} + h_2^{\sigma(l(i)-i+1)} \geq 1$, $i = \{1, \dots, l\}$.
4. $E(h_1) = E(h_1^c)$

with $l = \max\{l(h_1), l(h_2)\}$.

Remark 9. Note that the number of values of two HFEs may be different, $l(h_1) \neq l(h_2)$. Therefore, to carry out the computations correctly, both HFEs must have the same length l .

According to the axiomatic definition of entropy for HFEs, the following entropy formulas are defined⁶⁵:

$$E_1(h_1) = \frac{1}{l(h_1)(\sqrt{2}-1)} \sum_{i=1}^{l(h_1)} \left(\sin \frac{\pi(h_1^{\sigma(i)} + h_1^{\sigma(l(h_1)-i+1)})}{4} + \sin \frac{\pi(2 - h_1^{\sigma(i)} - h_1^{\sigma(l(h_1)-i+1)})}{4} - 1 \right) \quad (42)$$

$$E_2(h_1) = \frac{1}{l(h_1)(\sqrt{2}-1)} \sum_{i=1}^{l(h_1)} \left(\cos \frac{\pi(h_1^{\sigma(i)} + h_1^{\sigma(l(h_1)-i+1)})}{4} + \cos \frac{\pi(2 - h_1^{\sigma(i)} - h_1^{\sigma(l(h_1)-i+1)})}{4} - 1 \right) \quad (43)$$

$$E_3(h_1) = -\frac{1}{l(h_1) \ln 2} \sum_{i=1}^{l(h_1)} \left(\frac{h_1^{\sigma(i)} + h_1^{\sigma(l(h_1)-i+1)}}{2} \ln \frac{h_1^{\sigma(i)} + h_1^{\sigma(l(h_1)-i+1)}}{2} + \frac{2 - h_1^{\sigma(i)} + h_1^{\sigma(l(h_1)-i+1)}}{2} \ln \frac{2 - h_1^{\sigma(i)} + h_1^{\sigma(l(h_1)-i+1)}}{2} \right) \quad (44)$$

$$E_4(h_1) = \frac{1}{l(h_1)(2^{(1-s)t} - 1)} \sum_{i=1}^{l(h_1)} \left(\left(\left(\frac{h_1^{\sigma(i)} + h_1^{\sigma(l(h_1)-i+1)}}{2} \right)^s + \left(1 - \frac{h_1^{\sigma(i)} - h_1^{\sigma(l(h_1)-i+1)}}{2} \right)^s \right)^t - 1 \right) \tag{45}$$

where $t \neq 0, s \neq 1, s, t > 0$.

In Ref. 65, the cross-entropy was also introduced for HFEs and diverse relationships between entropy, cross-entropy, and similarity measures were studied.

Afterward, Farhadinia presented some counterexamples to prove that the definitions of entropy and cross-entropy introduced by Xu and Xia⁶⁵ cannot discriminate some HFEs, even though they are apparently different. Therefore, new entropies and relationships between entropy, similarity measures, and distance measures for HFSs are proposed in Ref. 66.

6. APPLICATIONS FOR HFSs

Once we have reviewed the concept of HFS, its extensions, different aggregation operators, and measures, our aim in this section is to show the applications based on HFSs.

The definition of HFS and its extensions have been applied mainly on decision-making, evaluation and clustering as shown in Table I.

- *Decision making:* HFSs are used by experts to provide their assessments or preferences over the set of criteria and alternatives defined in multicriteria decision making, group decision making, multiexpert multicriteria decision making, and decision support systems.
- *Evaluation:* The existence of real evaluation problems that deal with uncertain and vague information provoked by hesitation has driven to different proposals of evaluation using HFSs.
- *Clustering:* Clustering refers to a process that combines a set of objects into clusters according to the features of data. Usually, in real world, data used for clustering might be vague and imprecise. To manage this type of data, some clustering algorithms based on HFSs have been developed.

7. CRITICAL DISCUSSION AND NEW DIRECTIONS

The management of uncertain information in real-world problems is always difficult and complex. The concept of HFS facilitates dealing with uncertainty caused by hesitation. In this paper, we have shown different concepts, extensions and tools to handle hesitant information. Since, we would like to point out some critical comments and directions.

To manage hesitant information in real-world problems, different theoretical models have been introduced in the specialized literature. They are necessary for different applications as underlined in Table I. Nevertheless, there are some weaknesses

Table I. Applications based on the use of HFS

Applications	Papers	Representation	Year
Multicriteria decision making	Xia and Xu ¹⁷	HFS	2011
	Yu et al. ⁴³	HFS	2011
	Farhadinia ²⁰	HFS	2012
	Rodríguez et al. ¹⁶	HFLTS	2012
	Wei ⁴⁴	HFS	2012
	Xu and Xia ⁶⁵	HFS	2012
	Yu et al. ⁴⁶	HFS	2012
	Lee and Chen ³⁸	HFLTS	2013
	Liao and Xu ⁶⁷	HFS	2013
	Liu ⁵¹	HFS	2013
	Xu and Zhang ⁶⁸	HFS	2013
	Zhang and Wei ⁶⁹	HFS	2013
	Liu and Rodríguez ³⁷	HFLTS	2014
	Wei et al. ³⁹	HFLTS	2014
Zhou and Li ⁵⁵	HFS	2014	
Group decision making	Chen ¹²	IVHFS	2013
	Rodríguez et al. ³⁵	HFLTS	2013
	Zhu and Xu ⁴⁰	HFLTS	2014
Multiexpert multicriteria decision making	Zhu et al. ¹⁵	DHFS	2012
	Beg and Rashid ³⁶	HFLTS	2013
	Xia et al. ⁵⁰	HFS	2013
	Zhang and Xu ⁴²	HFLTS	2014
Decision support systems Evaluation processes	Qian et al. ¹³	GHFS	2013
	Yu ¹⁴	TFHFS	2013
	Yu and Zhang ⁴⁵	HFS	2013
Clustering algorithms	Sahu and Thakur ⁷⁰	HFS	2011
	Zhang and Xu ⁷¹	HFS	2012
	Chen et al. ⁶³	HFS	2013
	Farhadinia ⁶⁶	HFS	2013
	Zhang and Xu ⁷²	HFS	2014

that must be highlighted.

- The usefulness of some extensions of HFS is debatable. They must be justified from a theoretical or practical point of view and solve real problems with uncertainty.
- Too many aggregation operators for HFS have been defined without a clear justification of their necessity. Some of them are just applications of the extension principle. It is necessary a clear justification on their necessity and usefulness.
- Some concepts are not defined properly as we have shown in several remarks along the paper.
- Authors use different notation to define concepts, extensions, and tools to manage hesitant information.

Thus, some directions that should be considered to further research are the following ones:

- It is necessary to carry out developments on the theoretical models, but their needs must be justified.
- A new trend is the application of the theoretical models to real problems. Models that present solutions to problems that cannot be solved by approaches already defined. It

is necessary to justify the usefulness of the new models comparing them with previous approaches.⁷³

- A proposal of new aggregation operators must be justified, and their necessity and difference with the existing ones must be explained. A clear analysis of the usefulness and a comparative study of its use should be presented.
- New concepts must be defined clearly to be understood and used suitably.
- It is necessary to unify the notation to define concepts, extensions, and tools for HFSs.

8. CONCLUSIONS

Uncertainty usually appears in many real-world problems. Fuzzy sets and its extensions have provided successful results dealing with uncertainty in different problems. We have paid attention to one of them, HFS, that manages hesitant situations that often appear when the membership degree of an element to a set must be established. This new approach has attracted the attention of some researchers who have defined diverse concepts, extensions, aggregation operators, and measures to handle with hesitant information.

In spite of the usefulness of the HFSs evidenced by the applications in decision making, evaluation, and clustering, it is clear that the necessity of applying the concepts and tools is defined to real problems. We have pointed out some future directions and considerations that should be taken into account in the coming HFSs-based proposals, and we have unified the notation regarding HFSs.

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